Computational Information Games
A minitutorial Part II

Houman Owhadi

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(Computational Information Games)
**Question**

Can we design a linear solver with some degree of universality? (that could be applied to a large class of linear operators)

**Motivation**

There are (nearly) as many linear solvers as linear systems.  
Number of google scholar references to “linear solvers”: 447,000

**Not clear that this can be done**

“Of course no one method of approximation of a ‘linear operator’ can be universal.”


Arthur Sard  
(1909-1980)
Multigrid Methods


Multiresolution/Wavelet based methods

[Beylkin, Coifman, Rokhlin, 1992] [Engquist, Osher, Zhong, 1992]
[Alpert, Beylkin, Coifman, Rokhlin, 1993]
[Cohen, Daubechies, Feauveau. 1992]
[Bacry, Mallat, Papanicolaou. 1993]

• Linear complexity with smooth coefficients

Problem    Severely affected by lack of smoothness

\[
\begin{cases}
- \text{div}(a \nabla u) = g, & x \in \Omega, \\
    u = 0, & x \in \partial \Omega,
\end{cases}
\]
Robust/Algebraic multigrid

[Panayot - 2010]

Stabilized Hierarchical bases, Multilevel preconditioners

[Vassilevski - Wang, 1997, 1998]
[Panayot - Vassilevski, 1997]
[Chow - Vassilevski, 2003]
[Aksoyulu- Holst, 2010]

• Some degree of robustness
Low Rank Matrix Decomposition methods

Fast Multipole Method: [Greengard and Rokhlin, 1987]
Hierarchical Matrix Method: [Hackbusch et al., 2002]
[Bebendorf, 2008]:

\[ N \ln^{2d+8} N \] complexity

To achieve grid-size accuracy in \( L^2 \)-norm

Hierarchical numerical homogenization method

[H. Owhadi, Multigrid with rough coefficients and Multiresolution operator decomposition from Hierarchical Information Games. SIAM Review, 2017]

First Solve \( \mathcal{O}(N \ln^{3d} N) \) Subsequent solves \( \mathcal{O}(N \ln^{d+1} N) \)

To achieve grid-size accuracy in \( H^1 \)-norm
Sparse matrix Laplacians

Sparsified Cholesky and Multigrid Solvers for Connection Laplacians: [Kyng, Lee, Peng, Sachdeva, Spielman , 2016]

Approximate Gaussian Elimination: [Kyng and Sachdeva, 2016]

$N \text{ polylog}(N)$ complexity

Structured sparse matrices (SDD matrices)

Graph sparsification: [Spielman and Teng , 2004]

Diagonally dominant linear systems: [Spielman and Teng , 2014]

[Koutis, Miller, Gary and Peng , 2014]

[Cohen, Kyng, Miller, Pachocki, Peng, Rao, and Xu, 2014]

[Kelner, Orecchia, Sidford, Zhu, 2013]
The problem

\( \mathcal{T} \): Continuous linear bijection

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\mathcal{T}} & \mathcal{B}^* \\
\end{array}
\]

We want to approximate \( \mathcal{T}^{-1} \) and all its eigen-subspaces in near-linear complexity

For \( u, v \in \mathcal{B} \),

- \( [\mathcal{T}u, v] = [\mathcal{T}v, u] \),
- \( [\mathcal{T}u, u] \geq 0 \)

\[
\|u\|^2 := [\mathcal{T}u, u]
\]

\((\mathcal{B}, \| \cdot \|)\): separable Banach space
Example

\[
\begin{aligned}
- \operatorname{div}(a \nabla u) &= g, \quad x \in \Omega, \\
\quad u &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]

\[\mathcal{T} = - \operatorname{div}(a \nabla \cdot)\]

\[
(H^1_0(\Omega), \| \cdot \|_{H^1_0(\Omega)}) \xrightarrow{- \operatorname{div}(a \nabla \cdot)} (H^{-1}(\Omega), \| \cdot \|_{H^{-1}(\Omega)})
\]

\[\mathcal{B} := H^1_0(\Omega)\]

\[\|u\|^2 := \int_{\Omega} (\nabla u)^T a \nabla u\]
Example

$\mathcal{L} u = g$

$\mathcal{L}$: arbitrary continuous linear bijection

$(H_0^s(\Omega), \| \cdot \|_{H_0^s(\Omega)}) \xrightarrow{\mathcal{L}} (H^{-s}(\Omega), \| \cdot \|_{H^{-s}(\Omega)})$

$\mathcal{L}$: Symmetric and positive

- $[\mathcal{L} u, v] = [\mathcal{L} v, u]$
- $[\mathcal{L} u, u] \geq 0$

$\mathcal{B} := H_0^s(\Omega)$

$\mathcal{T} = \mathcal{L}$

$\| u \|^2 := [\mathcal{L} u, u]$
Example

\[ \mathcal{L}u = g \iff \mathcal{L}^* \mathcal{L}u = \mathcal{L}^* g \]

\( \mathcal{L} \): arbitrary continuous linear bijection

\[
(H_0^s(\Omega), \| \cdot \|_{H_0^s(\Omega)}) \xrightarrow{\mathcal{L}} (L^2(\Omega), \| \cdot \|_{L^2(\Omega)})
\]

\[ \mathcal{B} := H_0^s(\Omega) \]

\[ \mathcal{T} = \mathcal{L}^* \mathcal{L} \]

\[ \| u \| := \| \mathcal{L}u \|_{L^2(\Omega)} \]
Example

\[ \mathbf{A} \mathbf{x} = \mathbf{b} \]

A: \( N \times N \) symmetric positive definite matrix

\( \mathcal{B} := \mathbb{R}^N \)

\( \mathcal{T} = \mathbf{A} \)

\[ \|\mathbf{x}\|^2 := \mathbf{x}^T \mathbf{A} \mathbf{x} \]
Example

\[
Ax = b \quad \leftrightarrow \quad A^T Ax = A^T b
\]

\(A: \ N \times N\) invertible matrix

\(\mathcal{B} := \mathbb{R}^N\)

\(\mathcal{T} = A^T A\)

\[
\|x\|^2 := \|Ax\|^2
\]


\[ \mathcal{B} \xrightarrow{\mathcal{T}} \mathcal{B}^* \]

\[ \|u\|^2 := [\mathcal{T} u, u] \]

**Hierarchy of measurement functions**

\[ \phi^{(k)}_i \in \mathcal{B}^* \text{ with } k \in \{1, \ldots, q\} \]

\[ \phi^{(k)}_i = \sum_j \pi^{(k,k+1)}_{i,j} \phi^{(k+1)}_j \]

Example

$\mathcal{B} = H_0^s(\Omega)$

$\phi_i^{(k)}$: Weighted indicator functions of a hierarchical nested partition of $\Omega$ of resolution $2^{-k}$
Example \( \mathcal{B} = H_0^s(\Omega) \)

\((\phi_{i,\alpha}^{(k)})_{\alpha \in \mathcal{A}}\): orthonormal basis functions of \( \mathcal{P}_{s-1}(\tau_{i}^{(k)}) \)

\( \mathcal{P}_{s-1}(\tau_{i}^{(k)})\): polynomials of degree at most \( s - 1 \)


Example  \[ \mathcal{B} = H_0^s(\Omega) \quad s > d/2 \]

\[ \phi_i^{(k)} : \text{Subsampled delta Dirac functions} \]


Player I

Chooses

\( u \in B \)

Player II

Sees [\( \phi_i^{(k)}, u \)], \( i \in I_k \)

Must predict

\( u \) and [\( \phi_j^{(k+1)}, u \)], \( j \in I_{k+1} \)
Example

Player I

Chooses

\( u \in H^1_0(\Omega) \)

Player II

Sees \( \{ \int_{\Omega} u \phi_i^{(k)}, i \in I_k \} \)

Must predict

\( u \) and \( \{ \int_{\Omega} u \phi_j^{(k+1)}, j \in I_{k+1} \} \)
Player II’s bets

\[ u^{(k)} := \mathbb{E} \left[ \xi \mid \phi^{(k)}_i, \xi \right] = [\phi^{(k)}_i, u], \ i \in \mathcal{I}_k \]

\[ \mathcal{F}^{(k)} = \sigma([\phi^{(k)}_i, \xi], \ i \in \mathcal{I}_k) \]

\[ \xi^{(k)} = \mathbb{E}[\xi \mid \mathcal{F}^{(k)}] \]

\( \xi^{(k)} \): Martingale

\( \xi^{(k)} \): Converging a.s.

\( \xi^{(k+1)} - \xi^{(k)} \): Uncorrelated (therefore independent)
Example

\[ B = H^1_0(\Omega) \]

\[ \|u\|^2 = \int_{\Omega} (\nabla u)^T a \nabla u \]

\[
\begin{cases}
  - \text{div}(a \nabla u) = g, & x \in \Omega, \\
  u = 0, & x \in \partial \Omega,
\end{cases}
\]
Accuracy of the recovery

\[ \| u - u^{(k)} \| \leq \frac{H^k}{\lambda_{\min}(a)} \| g \|_{L^2(\Omega)} \]

\[ \phi_i^{(k)} = 1_{\tau_i^{(k)}} \quad \text{diam}(\tau_i^{(k)}) \leq H^k \]
\[ \begin{cases} - \text{div}(a \nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial \Omega, \end{cases} \quad g \in C^\infty(\Omega) \]

Energy content

If r.h.s. is regular we don’t need to compute all subbands
\[
\begin{align*}
\left\{ \begin{array}{l}
-\text{div}(a\nabla u) = g, \quad x \in \Omega, \\
\quad u = 0, \quad x \in \partial\Omega,
\end{array} \right.
\end{align*}
\]

\[g = \delta(x - x_0)\]
\[ u^{(k)} = \sum_i [\phi_i^{(k)}, u] \psi_i^{(k)} \]

Gamblets

\[ \psi_i^{(k)} = \mathbb{E}[\xi | [\phi_l^{(k)}, \xi] = \delta_{i,l}, l \in \mathcal{I}_k] \]
Example

\[ \mathcal{B} = H^1_0(\Omega) \]

\[ \|u\|^2 = \int_{\Omega} (\nabla u)^T a \nabla u \]

\[ \phi^{(k-1)}_i = \sum_j \pi_{i,j}^{(k-1,k)} \phi^{(k)}_j \]

\[ \pi_{i,j}^{(1,2)} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\( \phi_i^{(1)} \)
\( \phi_i^{(2)} \)
\( \phi_i^{(3)} \)
\( \phi_i^{(4)} \)
\( \phi_i^{(5)} \)
\( \phi_i^{(6)} \)
Gamblets

\[ \psi_i^{(1)} \]

\[ \psi_i^{(2)} \]

\[ \psi_i^{(3)} \]

\[ \psi_i^{(4)} \]

\[ \psi_i^{(5)} \]

\[ \psi_i^{(6)} \]

\[ \psi_i^{(7)} \]
Gamblets are nested

\[
\psi_{i}^{(k)} = \sum_{j} R_{i,j}^{(k,k+1)} \psi_{j}^{(k+1)}
\]

\[
\psi_{i}^{(k)} = \mathbb{E} \left[ \mathbb{E} \left[ \xi \mid F_{k+1} \right] \bigg| \phi_{l}^{(k)}, \xi \right] = \delta_{i,l}, \; l \in \mathcal{I}_{k}
\]

\[
\mathbb{E} \left[ \xi \mid F_{k+1} \right] = \sum_{j} [\phi_{j}^{(k+1)}, \xi] \psi_{j}^{(k+1)}
\]

\[
R_{i,j}^{(k,k+1)} = \mathbb{E} \left[ [\phi_{j}^{(k+1)}, \xi] [\phi_{l}^{(k)}, \xi] = \delta_{i,l}, \; l \in \mathcal{I}_{k} \right]
\]

Interpolation/Prolongation operator
Player I

Chooses

\[ u \in B \]

Player II

Sees \[ [\phi_i^{(k)}, u], \; i \in \mathcal{I}_k \]

Must predict

\[ [\phi_j^{(k+1)}, u], \; j \in \mathcal{I}_{k+1} \]

Optimal bet of Player II

on the value of \[ [\phi_j^{(k+1)}, u] \]

\[ \sum_i [\phi_i^{(k)}, u] R_{i,j}^{(k,k+1)} \]
Example

\[ \mathcal{B} = H^1_0(\Omega) \]

\[
\begin{align*}
R_{i,j}^{(k)} &= \mathbb{E} \left[ \int_\Omega \xi(y) \phi_j^{(k+1)}(y) \, dy \bigg| \int_\Omega \xi(y) \phi_l^{(k)}(y) \, dy = \delta_{i,l}, \ l \in \mathcal{I}_k \right] \\
\end{align*}
\]

Your best bet on the value of \( \int_{\mathcal{T}_{j}^{(k+1)}} u \) given the information that \( \int_{\mathcal{T}_{i}^{(k)}} u = 1 \) and \( \int_{\mathcal{T}_l} u = 0 \) for \( l \neq i \)

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]
\[ \mathcal{B} \xrightarrow{\mathcal{T}} \mathcal{B}^* \]

\[ \| u \|^2 := [\mathcal{T}u, u] \]

**Hierarchy of measurement functions**

\[ \phi_i^{(k)} \in \mathcal{B}^* \text{ with } k \in \{1, \ldots, q\} \]

\[ \phi_i^{(k)} = \sum_j \pi_{i,j}^{(k,k+1)} \phi_j^{(k+1)} \]

**Hierarchy of gamblets**

\[ \psi_i^{(k)} = \sum_{j \in \mathcal{I}^{(k)}} \Theta_{i,j}^{(k), -1} \mathcal{T}^{-1} \phi_j^{(k)} \]

\[ \Theta_{i,j}^{(k)} := [\phi_i^{(k)}, \mathcal{T}^{-1} \phi_j^{(k)}] \]
Biorthogonal system

\[ [\phi_j^{(k)}, \psi_i^{(k)}] = \delta_{i,j} \]

\[ \mathcal{V}^{(k)} := \text{span}\{\psi_i^{(k)} \mid i \in \mathcal{I}^{(k)}\} \]

Theorem

The \( \langle \cdot, \cdot \rangle \) orthogonal projection of \( u \in \mathcal{B} \) onto \( \mathcal{V}^{(k)} \) is

\[ u^{(k)} = \sum_{i \in \mathcal{I}^{(k)}} [\phi_i^{(k)}, u] \psi_i^{(k)} \]
Measurement functions are nested

\[ \phi_i^{(k)} = \sum_j \pi_{i,j}^{(k,k+1)} \phi_j^{(k+1)} \]

Gamblets are nested

\[ \psi_i^{(k)} = \sum_{j \in \mathcal{I}^{(k+1)}} R_{i,j}^{(k,k+1)} \psi_j^{(k+1)} \]

Orthogonalized gamblets

\[ \chi_i^{(k)} := \sum_{j \in \mathcal{I}^{(k)}} W_{i,j}^{(k)} \psi_j^{(k)} \]

For \( k \geq 2 \), \( W^{(k)} \): \( \mathcal{J}^{(k)} \times \mathcal{I}^{(k)} \) matrix such that \( \text{Img}(W^{(k)}, T) = \text{Ker}(\pi^{(k-1,k)}) \) and \( W^{(k)}(W^{(k)})^T = J^{(k)} \).
\[ \chi_{i}^{(k)} := \sum_{j \in \mathcal{I}^{(k)}} W_{i,j}^{(k)} \psi_{j}^{(k)} \]
Operator adapted MRA

\[ \mathcal{V}(k) := \text{span}\{\psi_i^{(k)} \mid i \in I^{(k)}\} \]

\[ \mathcal{W}(k) := \text{span}\{\chi_i^{(k)} \mid i \in I^{(k)}\} \]

**Theorem**

\[ \mathcal{V}(k) = \mathcal{V}(k-1) \oplus \mathcal{W}(k) \]

\[ \mathcal{B} = \mathcal{V}(1) \oplus \mathcal{W}(2) \oplus \mathcal{W}(3) \oplus \ldots \]

\[ u^{(k)} - u^{(k-1)}: \text{ The } \langle \cdot, \cdot \rangle \text{ orthogonal projection of } u \in \mathcal{B} \text{ onto } \mathcal{W}^{(k)} \]
Theorem

\[ u = u^{(1)} + \cdots + (u^{(k)} - u^{(k-1)}) + \cdots \]

\[ u^{(k)} - u^{(k-1)} = \sum_{i \in \mathcal{I}^{(k)}} w_i^{(k)} \chi_i^{(k)} \]

\[ B^{(k)} w^{(k)} = g^{(k)} \]

\[ g_i^{(k)} = [g, \chi_i^{(k)}] \quad B_{i,j}^{(k)} = \langle \chi_i^{(k)}, \chi_j^{(k)} \rangle \]
\[
\begin{align*}
- \text{div}(a \nabla u) &= g, \quad x \in \Omega, \\
\quad u &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

\[g \in C^\infty(\Omega)\]

If r.h.s. is regular we don’t need to compute all subbands
\[
\begin{align*}
\begin{cases}
- \text{div}(a \nabla u) &= g, & x \in \Omega, \\
u &= 0, & x \in \partial \Omega, \\
g &= \delta(x - x_0)
\end{cases}
\end{align*}
\]
Operator adapted wavelets

First Generation Wavelets: Signal and imaging processing
- [Mallat, 1989]
- [Daubechies, 1990]
- [Coifman, Meyer, and Wickerhauser, 1992]

First Generation Operator Adapted Wavelets (shift and scale invariant)
- [Cohen, Daubechies, Feauveau. Biorthogonal bases of compactly supported wavelets. 1992]
- [Beylkin, Coifman, Rokhlin, 1992]
- [Engquist, Osher, Zhong, 1992]
- [Alpert, Beylkin, Coifman, Rokhlin, 1993]
- [Jawerth, Sweldens, 1993]
- [Dahlke, Weinreich, 1993]
- [Bacry, Mallat, Papanicolaou. 1993]
- [Bertoluzza, Maday, Ravel, 1994]
- [Vasilyev, Paolucci, 1996]
- [Dahmen, Kunoth, 2005]
- [Stevenson, 2009]

Lazy wavelets (Multiresolution decomposition of solution space)
- [Yserentant. Multilevel splitting, 1986]
- [Bank, Dupont, Yserentant. Hierarchical basis multigrid method. 1988]
Operator adapted wavelets

Second Generation Operator Adapted Wavelets

[Sweldens. The lifting scheme, 1998] [Dorobantu - Engquist. 1998]
[Barinka, Barsch, Charton, Cohen, Dahlke, Dahmen, Urban, 2001]
[Cohen, Dahmen, DeVore, 2001] [Chiavassa, Liandrat, 2001]
[Dahmen, Kunoth, 2005] [Schwab, Stevenson, 2008]
[Sudarshan, 2005] [Engquist, Runborg, 2009] [Yin, Liandrat, 2016]

We want

1. Scale-orthogonal wavelets with respect to operator scalar product (leads to block-diagonalization)
2. Operator to be well conditioned within each subband
3. Wavelets need to be localized (compact support or exp. decay)
Eigenspace adapted MRA

\[ A^{(k)}_{i,j} = \langle \psi_i^{(k)} , \psi_j^{(k)} \rangle \quad B^{(k)}_{i,j} = \langle \chi_i^{(k)} , \chi_j^{(k)} \rangle \]

**Theorem** Under regularity of measurement functions

\[ \frac{1}{C} H^{-2(k-1)} J^{(k)} \leq B^{(k)} \leq C H^{-2k} J^{(k)} \]

\[ \text{Cond}(B^{(k)}) \leq C H^{-2} \]

\[ \frac{1}{C} I^{(1)} \leq A^{(1)} \leq C H^{-2} I^{(1)} \]

\[ \text{Cond}(A^{(1)}) \leq C H^{-2} \]
High contrast

\[
\log_{10} \left( \frac{\lambda_{\text{max}}(A^{(k)})}{\lambda_{\text{min}}(A^{(k)})} \right)
\]

\[
\log_{10} \left( \frac{\lambda_{\text{max}}(B^{(k)})}{\lambda_{\text{min}}(B^{(k)})} \right)
\]

Low contrast

\[
\log_{10} \left( \frac{\lambda_{\text{max}}(A^{(k)})}{\lambda_{\text{min}}(A^{(k)})} \right)
\]

\[
\log_{10} \left( \frac{\lambda_{\text{max}}(B^{(k)})}{\lambda_{\text{min}}(B^{(k)})} \right)
\]
Wannier functions

[Wannier. Dynamics of band electrons in electric and magnetic fields. 1962]
[Kohn. Analytic properties of Bloch waves and Wannier functions, 1959]
[E, Tiejun, Jianfeng. Localized bases of eigensubspaces and operator compression, 2010]
[Vidvuds, Lai, Caflisch, Osher, Compressed modes for variational problems in mathematics and physics, 2013]

[Owhadi, Multiresolution operator decomposition, SIREV 2017]
[Owhadi, Zhang, gamblets for hyperbolic and parabolic PDEs, 2016]
[Hou, Qin, Zhang, A sparse decomposition of low rank symmetric positive semi-definite matrices, 2016]
[Hou, Zhang, Sparse operator compression of elliptic operators, 2017]
Regularity Conditions

For some $H \in (0, 1)$ and $C_\Phi > 0$

1. $|x| \leq C_\Phi H^{-k} \| \phi \|_*$
   for $\phi \in \{ \sum_{i \in I(k)} x_i \phi_i^{(k)} \}$

2. $\| \phi \|_* \leq C_\Phi H^k |x|$
   for $\phi \in \{ \sum_{i \in I(k+1)} x_i \phi_i^{(k+1)} \mid x \in \text{Ker}(\pi^{(k,k+1)}) \}$

Conditions are covariant under norm equivalence
Example

\[ T = \mathcal{L} \]

\[ (H^s_0(\Omega), \| \cdot \|_{H^s_0(\Omega)}) \xrightarrow{\mathcal{L}} (H^{-s}(\Omega), \| \cdot \|_{H^{-s}(\Omega)}) \]

Regularity Conditions

For some \( H \in (0, 1) \) and \( C_s > 0 \)

1. \( |x| \leq C_s H^{-k} \| \phi \|_{H^{-s}(\Omega)} \)
   
   for \( \phi \in \{ \sum_{i \in I(k)} x_i \phi_i^{(k)} \} \)

2. \( \| \phi \|_{H^{-s}(\Omega)} \leq C_s H^k |x| \)
   
   for \( \phi \in \{ \sum_{i \in I(k+1)} x_i \phi_i^{(k+1)} \mid x \in \text{Ker}(\pi^{(k,k+1)}) \} \)
Example

\( s = 1 \)

\[ H = \frac{1}{2} \]

\( \phi_i^{(k)} \): Weighted indicator functions of a hierarchical nested partition of \( \Omega \) of resolution \( 2^{-k} \)
Example \[ s \geq 2 \quad H = \frac{1}{2^s} \]

\[(\phi_{i,\alpha}^{(k)})_{\alpha \in \mathcal{A}}: \text{orthonormal basis functions of } \mathcal{P}_{s-1}(\tau_i^{(k)})\]

\[\mathcal{P}_{s-1}(\tau_i^{(k)}): \text{polynomials of degree at most } s - 1\]

\[\tau_i^{(k)}: \text{Hierarchical nested partition of } \Omega \text{ of resolution } 2^{-k}\]


Example

\[ s \geq 2 \]

\[ H = \frac{1}{2^s} \]


\( \phi_i^{(k)} \): Weighted indicator functions of a hierarchical nested partition of \( \Omega \) of resolution \( 2^{-k} \)

\[ s > \frac{d}{2} \]

\( \phi_i^{(k)} \): Subsampled delta Dirac functions
Example \( \mathcal{B} := \mathbb{R}^N \) \[ \| x \|^2 := x^T A x \]

\( A: N \times N \) symmetric \[ \| x \|_*^2 := x^T A^{-1} x \]

positive definite matrix \[ \| x \|_0^2 := x^T x \]

\( \phi_i^{(q)} = e_i \)

\( \pi(k, k+1) (\pi(k, k+1))^T = I(k) \)

Regularity Conditions \( \pi(k, q) = \pi(k, k+1) \ldots \pi(q-1, q) \)

For some \( H \in (0, 1) \) and \( C > 0 \)

1. \[ \frac{1}{C \sqrt{\lambda_{\text{min}}(A)}} H^k \leq \inf_{x \in \text{Img}(\pi(q, k))} \frac{\sqrt{x^T A^{-1} x}}{|x|} \]

2. \[ \sup_{x \in \text{Ker}(\pi(k, q))} \frac{\sqrt{x^T A^{-1} x}}{|x|} \leq \frac{C}{\sqrt{\lambda_{\text{min}}(A)}} H^k \]

Conditions are covariant under quadratic form equivalence
Regularity Conditions on Primal Space

For some $H \in (0, 1)$ and $C_{\Phi} > 0$

1. $C_{\Phi}^{-1} H^k \frac{\sqrt{x^T A x}}{\lambda_{\min}(A)} \leq |x|$
   for $x \in \text{Img}(\pi^{(q,k)})$ and $k \in \{1, \ldots, q\}$.

2. $\inf_{y \in \mathbb{R}^{(k)}} |z - \pi^{(q,k)} y| \leq C_{\Phi} H^k \frac{\sqrt{z^T A z}}{\lambda_{\min}(A)}$
   for $z \in \mathbb{R}^N$ and $k \in \{1, \ldots, q\}$. 
1: For $i \in \mathcal{I}^{(q)}$, $\psi_i^{(q)} = \varphi_i$
2: For $i \in \mathcal{I}^{(q)}$, $g_i^{(q)} = [g, \psi_i^{(q)}]$
3: For $i, j \in \mathcal{I}^{(q)}$, $A_{i,j}^{(q)} = \langle \psi_i^{(q)}, \psi_j^{(q)} \rangle$
4: for $k = q$ to 2 do
5: $B^{(k)} = W^{(k)} A^{(k)} W^{(k)} T$
6: $w^{(k)} = B^{(k)},-1 W^{(k)} g^{(k)}$
7: For $i \in \mathcal{J}^{(k)}$, $\chi_i^{(k)} = \sum_{j \in \mathcal{I}^{(k)}} W_{i,j}^{(k)} \psi_j^{(k)}$
8: $u^{(k)} - u^{(k-1)} = \sum_{i \in \mathcal{J}^{(k)}} w_i^{(k)} \chi_i^{(k)}$
9: $D^{(k,k-1)} = -B^{(k)},-1 W^{(k)} A^{(k)} \bar{\pi}^{(k,k-1)}$
10: $R^{(k-1,k)} = \bar{\pi}^{(k-1,k)} + D^{(k-1,k)} W^{(k)}$
11: $A^{(k-1)} = R^{(k-1,k)} A^{(k)} R^{(k,k-1)}$
12: For $i \in \mathcal{I}^{(k-1)}$, $\psi_i^{(k-1)} = \sum_{j \in \mathcal{I}^{(k)}} R_{i,j}^{(k-1,k)} \psi_j^{(k)}$
13: $g^{(k-1)} = R^{(k-1,k)} g^{(k)}$
14: end for
15: $U^{(1)} = A^{(1)},-1 g^{(1)}$
16: $u^{(1)} = \sum_{i \in \mathcal{I}^{(1)}} U_i^{(1)} \psi_i^{(1)}$
17: $u = u^{(1)} + (u^{(2)} - u^{(1)}) + \cdots + (u^{(q)} - u^{(q-1)})$
Fast Gamblet Transform obtained by truncation/localization

**Complexity Theorem**

\[ N = \text{Card}(\mathcal{I}^{(q)}) \]

- \( N \log^{3d}(N) \): Computation of all gamblets
- \( N \log^{d+1}(N) \): Gamblet transform/solve of \( u \in \mathcal{B} \) to accuracy \( H^q \) in \( \| \cdot \| \) norm

Based on exponential decay of gamblets and locality of the operator

- \( d \): Hausdorff dimension of \( d^A \).
- \( d^A \): Graph distance of \( A \) on \( \mathcal{I}^{(q)} \)
- \( A_{i,j} := \langle \varphi_i, \varphi_j \rangle \), stiffness matrix of the operator

\[ \text{Card}\{j|d^A_{i,j} \leq r\} \leq C r^d \]
Localization of Gamblets

\[ \mathcal{B} := H^s_0(\Omega) \quad \|u\|^2 := [Lu, u] \]

\[ \psi_i = \mathbb{E} \left[ \xi \mid [\phi_j, \xi] = \delta_{i,j} \text{ for } j \in \mathcal{I} \right] \]

\[ \phi_i = 1_{\tau_i} \]
Sparsity of the precision matrix

\[ \Theta_{i,j} = \text{Cov} \left( [\phi_i, \xi], [\phi_j, \xi] \right) \]

\[ \Theta_{i,j}^{-1} = \langle \psi_i, \psi_j \rangle \]

\[ \Theta_{i,j}^{-1} = 0 \]

\[ \text{Cov} \left( [\phi_i, \xi], [\phi_j, \xi] | [\phi_l, \xi], l \neq i, j \right) = 0 \]
Localization problem in Numerical Homogenization

[Chu-Graham-Hou-2010] (limited inclusions)
[Efendiev-Galvis-Wu-2010] (limited inclusions or mask)
[Babuska-Lipton 2010] (local boundary eigenvectors)
[Owhadi-Zhang 2011] (localized transfer property)
[Malqvist-Peterseim 2012] Local Orthogonal Decomposition
[Owhadi-Zhang-Berlyand 2013] (Rough Polyharmonic Splines)
[ A. Gloria, S. Neukamm, and F. Otto, 2015] (quantification of ergodicity)
[Hou and Liu,DCDS-A, 2016]  [Chung-Efendiev-Hou, JCP 2016]
[Owhadi, Multiresolution operator decomposition, SIREV 2017]
[Owhadi, Zhang, gamblets for hyperbolic and parabolic PDEs, 2016]
[Hou, Qin, Zhang, 2016]  [Hou, Zhang, 2017]
[Hou and Zhang, 2017]: Higher order PDEs (localization under strong ellipticity,
h sufficiently small, and higher order polynomials as measurement functions)
[Kornhuber, Peterseim, Yserentant, 2016]: Subspace decomposition

Subspace decomposition/correction and Schwarz iterative methods

[J. Xu, 1992]: Iterative methods by space decomposition and subspace correction
[Griebel-Oswald, 1995]: Schwarz algorithms
Example

\[ \mathcal{B} := H_0^s(\Omega) \quad \quad \|u\|^2 := [\mathcal{L}u, u] \]

\(\mathcal{L}\): arbitrary continuous positive symmetric linear bijection

\[ (H_0^s(\Omega), \| \cdot \|_{H_0^s(\Omega)}) \xrightarrow{\mathcal{L}} (H^{-s}(\Omega), \| \cdot \|_{H^{-s}(\Omega)}) \]

\(\mathcal{L}\) is local \( \langle u, v \rangle = 0 \) if \( u \) and \( v \) have disjoint supports
Examples

\[ \phi_i = \frac{1_{T_i}}{\sqrt{|T_i|}}. \]

\[ \phi_i = \delta(\cdot - x_i), \quad (s > \frac{d}{2}) \]

\[ (\phi_i, \alpha)_{\alpha \in \mathbb{N}} \text{ forms an orthonormal basis of } \mathcal{P}_{s-1}(T_i) \]
Theorem

\[ \| \psi_{i, \alpha} - \psi_{i, \alpha}^n \|_{H^s_0(\Omega)} \leq Ce^{-n/C} \]
Condition for localization

For $\varphi \in H^{-s}(\Omega)$

$$C_{\text{min}} \leq \frac{\sum_i \inf_{\phi \in \Phi} \| \varphi - \phi \|^2_{H^{-s}(\Omega_i)}}{\inf_{\phi \in \Phi} \| \varphi - \phi \|^2_{H^{-s}(\Omega)}} \leq C_{\text{max}}$$

$$\Phi = \{ \phi_{i,\alpha} \mid (i, \alpha) \in \mathbb{I} \times \mathbb{N} \}$$
Theorem

Assume that there exists a constant $C_0$ such that $|\mathcal{N}| \leq C_0$,

- $\| D^t f \|_{L^2(\Omega)} \leq C_0 h^{s-t} \| f \|_{H^s_0(\Omega)}$ for $t \in \{0, 1, \ldots, s\}$, for $f \in H^s_0(\Omega)$ such that $[\phi_i, \alpha, f] = 0$ for $(i, \alpha) \in \mathcal{I} \times \mathcal{N}$,

- $\sum_{i \in \mathcal{I}, \alpha \in \mathcal{N}} [\phi_i, \alpha, f]^2 \leq C_0 (\| f \|_{L^2(\Omega)}^2 + h^{2s} \| f \|_{H^s_0(\Omega)}^2)$, for $f \in H^s_0(\Omega)$, and

- $|x|^2 \leq C_0 h^{-2s} \| \sum_{\alpha \in \mathcal{N}} x^H \phi_i, \alpha \|_{H^{-s}(\tau_i)}^2$, for $i \in \mathcal{I}$ and $x \in \mathbb{R}^\mathcal{N}$.

Then for $\varphi \in H^{-s}(\Omega)$

$$C_{\min} \leq \frac{\sum_i \inf_{\phi \in \Phi} \| \varphi - \phi \|_{H^{-s}(\Omega_i)}^2}{\inf_{\phi \in \Phi} \| \varphi - \phi \|_{H^{-s}(\Omega)}^2} \leq C_{\max}$$

Where $C_{\max}, C_{\min}$ depend only on $C_0, d, \delta$ and $s$
Banach space setting

\[ \mathcal{B} = \sum_{i \in I} \mathcal{B}_i \]

\[ \| \cdot \|_i \text{ and } \| \cdot \|_i,* \text{ norms induced by } \| \cdot \| \text{ on } \mathcal{B}_i \text{ and } \mathcal{B}_i^* \]

Condition for localization

For \( \varphi \in \mathcal{B}^* \)

\[ C_{\text{min}} \leq \frac{\sum_i \inf_{\phi \in \Phi} \| \varphi - \phi \|_{i,*}^2}{\inf_{\phi \in \Phi} \| \varphi - \phi \|_{*}^2} \leq C_{\text{max}} \]

\[ \Phi = \{ \phi_{i,\alpha} \mid (i, \alpha) \in I \times \mathbb{N} \} \]
Operator connectivity distance

\[ C: \mathcal{V} \times \mathcal{V} \text{ connectivity matrix} \]

\[ C_{i,j} = 1 \text{ if } \exists (\chi_i, \chi_j) \in \mathcal{B}_i \times \mathcal{B}_j \text{ s.t. } \langle \chi_i, \chi_j \rangle \neq 0 \]

\[ C_{i,j} = 0 \text{ otherwise} \]

\[ d: \text{ Graph distance on } \mathcal{V} \text{ induced by } C \]

\[ \psi_{i,\alpha}^n: \text{ Localization of } \psi_{i,\alpha} \text{ to } \mathcal{B}_i^n \]

\[ \mathcal{B}_i^n = \bigcup_{j: d(i,j) \leq n} \mathcal{B}_i \]

**Theorem**

Under localization conditions

\[ \| \psi_{i,\alpha} - \psi_{i,\alpha}^n \| \leq Ce^{-n/C} \]
Thank you


- Multigrid with gambllets. L. Zhang and H. Owhadi, 2017


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